

JACOB'S LADDERS AND CERTAIN ASYMPTOTIC MULTIPLICATIVE FORMULA FOR THE FUNCTION $|\zeta(\frac{1}{2} + it)|^2$

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ABSTRACT. In this paper it is proved that a mean-value of the product of some factors $|\zeta|^2$ is asymptotically equal to the product of the mean-values of $|\zeta|^2$, and this holds true for every fixed number of the factors.

1. INTRODUCTION

1.1. Let

$$(1.1) \quad Z(t) = e^{i\vartheta(t)} \zeta\left(\frac{1}{2} + it\right), \quad \vartheta(t) = -\frac{t}{2} \ln \pi + \operatorname{Im} \ln \Gamma\left(\frac{1}{4} + i\frac{t}{2}\right)$$

be the signal generated by the Riemann zeta-function. Hardy and Littlewood started to study the following integral in 1918

$$(1.2) \quad \int_0^T \left| \zeta\left(\frac{1}{2} + it\right) \right|^2 dt = \int_0^T Z^2(t) dt,$$

and they have derived the following formula (see [1], pp. 122, 151-156)

$$(1.3) \quad \int_0^T Z^2(t) dt \sim T \ln T, \quad T \rightarrow \infty.$$

In this direction the Titchmarsh-Kober-Atkinson (TKA) formula

$$(1.4) \quad \int_0^\infty Z^2(t) e^{-2\delta t} dt = \frac{c - \ln(4\pi\delta)}{2 \sin \delta} + \sum_{n=0}^N c_n \delta^n + \mathcal{O}(\delta^{N+\epsilon})$$

(see [5], p. 141) where c is the Euler's constant, remained as an isolated result for the period of 56 years. In our paper [2], we have discovered the nonlinear integral equation

$$(1.5) \quad \int_0^{\mu[x(T)]} Z^2(t) e^{-\frac{2}{x(T)}} dt = \int_0^T Z^2(t) dt,$$

where

$$\mu(y) \geq 7y \ln y, \quad \mu(y) \rightarrow y = \varphi_\mu(T) = \varphi(T),$$

in which the essence of the TKA formula (1.4) is inclosed. Namely, we have shown in [2] that the following infinite set of the almost-exact expressions of the Hardy-Littlewood integral (1.1) takes place

Key words and phrases. Riemann zeta-function.

$$(1.6) \quad \int_0^T Z^2(t) dt = \varphi_1(T) \ln \varphi_1(T) + (c - \ln 2\pi) \varphi_1(T) + c_0 + \mathcal{O}\left(\frac{\ln T}{T}\right),$$

$$\varphi_1(T) = \frac{1}{2} \varphi(T)$$

where $\varphi(T)$ (and, of course, also $\varphi_1(T)$) is the Jacob's ladder, i.e. an arbitrary solution to the nonlinear integral equation (1.5).

Remark 1. We have proved in the paper of reference [2] that except the asymptotic formula (1.3) possessing an unbounded error term (let us remark in this direction that the formulae of Ingham, Titchmarsh and Balasubramanian also possess the unbounded errors, comp. [2], Remark 2) there is an infinite family of almost exact representations (1.6) of the Hardy-Littlewood integral (1.3).

Remark 2. We can formulate the result (1.6) as follows: the Jacob's ladders $\varphi_1(t)$ are the asymptotic solutions of the transcendental equation

$$\int_0^T Z^2(t) dt = V(T) \ln V(T) + (c - \ln 2\pi) V(T) + c_0.$$

1.2. Next, we have proved (see [3], (8.3)) the following sixth-order formula

$$(1.7) \quad \int_T^{T+U_1} \left| \zeta\left(\frac{1}{2} + i\varphi_1(t)\right) \right|^4 \left| \zeta\left(\frac{1}{2} + it\right) \right|^2 dt \sim \frac{1}{2\pi^2} U_1 \ln^5 T,$$

$$U_1 = T^{\frac{7}{8}+2\epsilon}, \quad T \rightarrow \infty,$$

and (see [4], (2.5)) the following eight-order formula

$$(1.8) \quad \int_{\hat{T}}^{\widehat{T+U_2}} \left| \zeta\left(\frac{1}{2} + i\varphi_2(t)\right) \right|^4 \left| \zeta\left(\frac{1}{2} + it\right) \right|^4 dt \sim \frac{1}{4\pi^4} U_2 \ln^8 T, \quad T \rightarrow \infty$$

where

$$[T, T+U_2] = \varphi_2 \left\{ \left[\hat{T}, \widehat{T+U_2} \right] \right\}, \quad U_2 = T^{\frac{13}{14}+2\epsilon},$$

and φ_2 is the Jacob's ladder of the second order.

1.3. A motivation for the next step is in the well-known multiplicative formula

$$(1.9) \quad M\left(\prod_{k=1}^n X_k\right) = \prod_{k=1}^n M(X_k)$$

from the theory of probability, where X_k are independent random variables and M is the population mean.

We shall introduce, in connection with the formula (1.9), some new classes of non-local formulae for the function $|\zeta(\frac{1}{2} + it)|^2$. Namely, we shall show that there are functions

$$w_k(t), \quad k = 0, 1, \dots, n, \quad t \in [T, T+U]$$

such that the following formula

$$(1.10) \quad \begin{aligned} & \frac{1}{U} \int_T^{T+U} \prod_{k=0}^n \left| \zeta \left(\frac{1}{2} + iw_k(t) \right) \right|^2 dt \sim \\ & \sim \prod_{k=0}^n \frac{1}{w_k(T+U) - w_k(T)} \int_{w_k(T)}^{w_k(T+U)} \left| \zeta \left(\frac{1}{2} + it \right) \right|^2 dt, \quad T \rightarrow \infty \end{aligned}$$

holds true for every fixed $n \in \mathbb{N}$.

2. THE RESULT

Let

$$(2.1) \quad \begin{aligned} y &= \frac{1}{2} \varphi(t) = \varphi_1(t); \quad \varphi_1^0(t) = t, \quad \varphi_1^1(t) = \varphi_1(t), \\ \varphi_1^2(t) &= \varphi_1(\varphi_1(t)), \dots, \varphi_1^k(t) = \varphi_1(\varphi_1(\dots(\varphi_1(t)))) \dots, \dots, \\ t &\in [T, T+U], \end{aligned}$$

where $\varphi_1^k(t)$ denotes the k -th iteration of the Jacob's ladder

$$y = \varphi_1(t), \quad t \geq T_0[\varphi_1].$$

The following Theorem holds true.

Theorem. Let

$$(2.2) \quad U \in \left(0, \frac{T}{\ln^2 T} \right].$$

Then for every fixed $n \in \mathbb{N}$ we have

$$(2.3) \quad \begin{aligned} & \frac{1}{U} \int_T^{T+U} \prod_{k=0}^n \left| \zeta \left(\frac{1}{2} + i\varphi_1^k(t) \right) \right|^2 dt \sim \\ & \sim \prod_{k=0}^n \frac{1}{\varphi_1^k(T+U) - \varphi_1^k(T)} \int_{\varphi_1^k(T)}^{\varphi_1^k(T+U)} \left| \zeta \left(\frac{1}{2} + it \right) \right|^2 dt, \quad T \rightarrow \infty \end{aligned}$$

where

$$(2.4) \quad \varphi_1^k(t) \geq (1 - \epsilon)T, \quad k = 0, 1, \dots, n+1, \quad t \in [T, T+U].$$

Remark 3. Some nonlocal and non-linear interaction of the signals

$$\left| \zeta \left(\frac{1}{2} + it \right) \right|^2, \quad \left| \zeta \left(\frac{1}{2} + i\varphi_1^1(t) \right) \right|^2, \quad \dots, \quad \left| \zeta \left(\frac{1}{2} + i\varphi_1^n(t) \right) \right|^2$$

is expressed by the formula (2.3) and, simultaneously, a new art of the asymptotic independence of the partial functions

$$\left| \zeta \left(\frac{1}{2} + it \right) \right|^2, \quad t \in [\varphi_1^k(T), \varphi_1^k(T+U)], \quad k = 0, 1, \dots$$

is expressed by this formula.

Remark 4. We can put in the formula (2.3) for example

$$U = \frac{1}{T^2}, \quad U = \frac{2\pi}{\ln T} \sim t_{\nu+1} - t_\nu, \quad t_\nu \in \left[T, \frac{T}{\ln^2 T} \right], \dots$$

where $\{t_\nu\}$ is the Gram sequence, i.e. we can begin from the microscopic segments

$$[T, T+U] = \left[T, T + \frac{1}{T^2} \right], \quad \left[T, T + \frac{2\pi}{\ln T} \right], \dots$$

Remark 5. It is obvious that the formula (2.3) cannot be obtained in the theories of Balasubramanian, Heath-Brown and Ivic.

Remark 6. Every Jacob's ladder

$$\varphi_1(T) = \frac{1}{2}\varphi(t)$$

where φ_t is the exact solution of the nonlinear integral equation (1.5) is the asymptotic solutions of the following nonlinear *integro-iterations* equation

$$(2.5) \quad \begin{aligned} & \frac{1}{U} \int_T^{T+U} \prod_{k=0}^n \left| \zeta \left(\frac{1}{2} + ix^k(t) \right) \right|^2 dt = \\ & = \prod_{k=0}^n \frac{1}{x^k(T+U) - x^k(T)} \int_{x^k(T)}^{x^k(T+U)} \left| \zeta \left(\frac{1}{2} + it \right) \right|^2 dt \end{aligned}$$

of the new kind, where (comp. (2.1))

$$x^0(t) = t, \quad x^1(t) = x(t), \quad x^2(t) = x(x(t)), \quad \dots,$$

i.e. the function $x^k(t)$ is the k -th iteration of the function $x(t)$, $t \geq T_0[\varphi_1]$ (comp. (2.5) with [3], (11.1), (11.4), (11.6), (11.8) and [4], (2.6)).

3. PROOF OF THEOREM

3.1. We use the formulae (see [2], (6.2))

$$(3.1) \quad t - \varphi_1(t) \sim (1-c)\pi(t) \sim (1-c)\frac{t}{\ln t}, \quad t \rightarrow \infty.$$

Remark 7. The fundamental geometric property of the set of Jacob's ladders is expressed by the formula (3.1). Namely, the difference of the abscissa and the ordinate of the point $[t, \varphi_1(t)]$ of the curve $y = \varphi_1(t)$ is asymptotically equal to $(1-c)\pi(t)$.

We have (see (3.1))

$$(3.2) \quad \begin{aligned} & \varphi_1^1(t) - \varphi_1^2(t) \sim (1-c) \frac{\varphi_1^1(t)}{\ln \varphi_1^1(t)}, \\ & \varphi_1^2(t) - \varphi_1^3(t) \sim (1-c) \frac{\varphi_1^2(t)}{\ln \varphi_1^2(t)}, \\ & \vdots \\ & \varphi_1^n(t) - \varphi_1^{n+1}(t) \sim (1-c) \frac{\varphi_1^n(t)}{\ln \varphi_1^n(t)}, \quad t \in [T, T+U] \end{aligned}$$

for arbitrary fixed $n \in \mathbb{N}$ and (see (3.1), (3.2))

$$(3.3) \quad \begin{aligned} t &\sim \varphi_1^1(t) \sim \varphi_1^2(t) \sim \dots \sim \varphi_1^{n+1}(t), \quad t \rightarrow \infty, \\ t &> \varphi_1^1(t) > \varphi_1^2(t) > \dots > \varphi_1^{n+1}(t). \end{aligned}$$

Next we have (see (3.2), (3.3))

$$(3.4) \quad \begin{aligned} \varphi_1^1(t) - \varphi_1^2(t) &\sim (1-c) \frac{t}{\ln t}, \\ \varphi_1^2(t) - \varphi_1^3(t) &\sim (1-c) \frac{t}{\ln t}, \\ &\vdots \\ \varphi_1^n(t) - \varphi_1^{n+1}(t) &\sim (1-c) \frac{t}{\ln t}, \end{aligned}$$

and, consequently, by the addition of (3.1) and (3.4), we obtain

$$(3.5) \quad \begin{aligned} t - \varphi_1^{n+1}(t) &\sim (1-c)(n+1) \frac{t}{\ln t}, \\ \varphi_1^{n+1}(t) &\sim t \left\{ 1 - \frac{(1-c)(n+1)}{\ln t} \right\}, \quad 0 < 1-c < 1, \\ \varphi_1^{n+1}(t) &> \left(1 - \frac{\epsilon}{2}\right) t \left\{ 1 - \frac{(1-c)(n+1)}{\ln t} \right\} > (1-\epsilon)t \geq (1-\epsilon)T. \end{aligned}$$

(see (2.2)), i.e. from (3.5) by (3.3) (the second line) the formula (2.4) follows. Especially, the following holds true (see (3.3), (3.5))

$$(3.6) \quad (1-\epsilon)T < \varphi_1^{n+1}(T) < T.$$

3.2. Let (see [3], (9.1), (9.2))

$$(3.7) \quad \tilde{Z}^2(t) = \frac{d\varphi_1(t)}{dt}, \quad \varphi_1(t) = \frac{1}{2}\varphi(t), \quad t \geq T_0[\varphi_1]$$

where

$$(3.8) \quad \begin{aligned} \tilde{Z}^2(t) &= \frac{Z^2(t)}{2\Phi'_\varphi[\varphi(t)]} = \frac{|\zeta(\frac{1}{2} + it)|^2}{\{1 + \mathcal{O}(\frac{\ln \ln t}{\ln t})\} \ln t}, \\ t &\in [T, T+U], \quad U \in \left(0, \frac{T}{\ln T}\right]. \end{aligned}$$

If we use the formula (3.7) for the iterations (2.1) we obtain

$$(3.9) \quad \prod_{k=0}^n \tilde{Z}^2[\varphi_1^k(t)] = \frac{d\varphi_1^1}{dt} \frac{d\varphi_1^2}{d\varphi_1^1} \dots \frac{d\varphi_1^n}{d\varphi_1^{n-1}} \frac{d\varphi_1^{n+1}}{d\varphi_1^n} = \frac{d\varphi_1^{n+1}}{dt}$$

by the rule for differentiation of a composite function. Next, by the integration of (3.8) we obtain

$$(3.10) \quad \int_T^{T+U} \prod_{k=0}^n \tilde{Z}^2[\varphi_1^k(t)] dt = \varphi_1^{n+1}(T+U) - \varphi_1^{n+1}(T).$$

3.3. It follows easily from (3.7) that

$$\begin{aligned} \int_{\varphi_1^k(T)}^{\varphi_1^k(T+U)} \tilde{Z}^2(t) dt &= \varphi_1(\varphi_1^k(T+U)) - \varphi_1(\varphi_1^k(T)) = \\ &= \varphi_1^{k+1}(T+U) - \varphi_1^{k+1}(T), \quad k = 0, 1, \dots, n, \end{aligned}$$

i.e.

$$\frac{1}{\varphi_1^k(T+U) - \varphi_1^k(T)} \int_{\varphi_1^k(T)}^{\varphi_1^k(T+U)} \tilde{Z}^2(t) dt = \frac{\varphi_1^{k+1}(T+U) - \varphi_1^{k+1}(T)}{\varphi_1^k(T+U) - \varphi_1^k(T)},$$

and, consequently,

$$(3.11) \quad \prod_{k=0}^n \frac{1}{\varphi_1^k(T+U) - \varphi_1^k(T)} \int_{\varphi_1^k(T)}^{\varphi_1^k(T+U)} \tilde{Z}^2(t) dt = \frac{\varphi_1^{n+1}(T+U) - \varphi_1^{n+1}(T)}{U}.$$

Hence, from (3.10), (3.11) we obtain the *exact* formula

$$(3.12) \quad \begin{aligned} \frac{1}{U} \int_T^{T+U} \prod_{k=0}^n \tilde{Z}^2[\varphi_1^k(t)] dt &= \\ &= \prod_{k=0}^n \frac{1}{\varphi_1^k(T+U) - \varphi_1^k(T)} \int_{\varphi_1^k(T)}^{\varphi_1^k(T+U)} \tilde{Z}^2(t) dt. \end{aligned}$$

3.4. First of all, we have (see (2.2), (3.1))

$$\begin{aligned} \left(1 - \frac{1}{2n+1}\right) (1-c) \frac{T}{\ln T} &< T+U - \varphi_1^1(T+U) < \left(1 + \frac{1}{2n+1}\right) (1-c) \frac{T}{\ln T}, \\ \left(1 - \frac{1}{2n+1}\right) (1-c) \frac{T}{\ln T} &< T - \varphi_1^1(T+U) < \left(1 + \frac{1}{2n+1}\right) (1-c) \frac{T}{\ln T}. \end{aligned}$$

Next, we have ($0 < 1-c < 1$)

$$\begin{aligned} |\{T+U - \varphi_1^1(T+U)\} - \{T - \varphi_1^1(T)\}| &< \\ &< \left(1 + \frac{1}{2n+1}\right) (1-c) \frac{T}{\ln T} - \left(1 - \frac{1}{2n+1}\right) (1-c) \frac{T}{\ln T} = \\ &= \frac{2}{2n+1} (1-c) \frac{T}{\ln T} < \frac{2}{2n+1} \frac{T}{\ln T}, \end{aligned}$$

i.e.

$$(3.13) \quad \varphi_1^1(T+U) - \varphi_1^1(T) - U < \frac{2}{2n+1} \frac{T}{\ln T},$$

and (see (2.2))

$$\begin{aligned}
0 < \varphi_1^1(T+U) - \varphi_1^1(T) &< \frac{2}{2n+1} \frac{T}{\ln T} + U \leq \frac{2}{2n+1} \frac{T}{\ln T} + \frac{T}{\ln^2 T} < \\
&< \frac{2}{2n+1} \frac{T}{\ln T} + \frac{1}{2n+1} \frac{T}{\ln T} = \frac{3}{2n+1} \frac{T}{\ln T}.
\end{aligned}$$

Hence (see (2.2))

$$(3.14) \quad U \leq \frac{T}{\ln^2 T} \Rightarrow \varphi_1^1(T+U) - \varphi_1^1(T) < \frac{3}{2n+1} \frac{T}{\ln T}.$$

Similarly, from the formula (see (3.4))

$$\varphi_1^1(t) - \varphi_1^2(t) \sim (1-c) \frac{t}{\ln t}, \quad t \rightarrow \infty$$

we obtain (comp. (3.13)) that (see (3.14))

$$\varphi_1^2(T+U) - \varphi_1^2(T) < \frac{2}{2n+1} \frac{T}{\ln T} + \varphi_1^1(T+U) - \varphi_1^1(T) < \frac{5}{2n+1} \frac{T}{\ln T}.$$

Next, if the estimate (the function φ_1^k is increasing)

$$(0 <) \varphi_1^k(T+U) - \varphi_1^k(T) < \frac{2k+1}{2n+1} \frac{T}{\ln T}$$

holds true then we obtain, by a similar way, that

$$\varphi_1^{k+1}(T+U) - \varphi_1^{k+1}(T) < \frac{2}{2n+1} \frac{T}{\ln T} + \frac{2k+1}{2n+1} \frac{T}{\ln T} < \frac{2(k+1)+1}{2n+1} \frac{T}{\ln T}.$$

Hence, the following estimate holds true

$$(3.15) \quad \varphi_1^k(T+U) - \varphi_1^k(T) < \frac{2k+1}{2n+1} \frac{T}{\ln T} \leq \frac{T}{\ln T}, \quad k = 1, \dots, n.$$

(Comp. the condition for U in (3.8)).

3.5. Let us remind that by (3.6) we have the following inequalities

$$(1-\epsilon)T < \varphi_1^{n+1}(T) < T+U, \quad U \in \left(0, \frac{T}{\ln^2 T}\right],$$

i.e. for every

$$T' \in (\varphi_1^{n+1}(T), T+U)$$

the following holds true

$$(3.16) \quad \ln T' = \ln T + \mathcal{O}(1), \quad T \rightarrow \infty.$$

Now, if we use the mean-value theorem in (3.12) we obtain (see (3.3), (3.8), (3.15), (3.16))

$$(3.17) \quad \int_T^{T+U} \prod_{k=0}^n \tilde{Z}^2[\varphi_1^k(t)] dt \sim \frac{1}{\ln^{n+1} T} \int_T^{T+U} \prod_{k=0}^n \left| \zeta \left(\frac{1}{2} + i\varphi_1^k(t) \right) \right|^2 dt,$$

$$\begin{aligned}
(3.18) \quad & \prod_{k=0}^n \frac{1}{\varphi_1^k(T+U) - \varphi_1^k(T)} \int_{\varphi_1^k(T)}^{\varphi_1^k(T+U)} \tilde{Z}^2(t) dt \sim \\
& \sim \frac{1}{\ln^{n+1} T} \prod_{k=0}^n \frac{1}{\varphi_1^k(T+U) - \varphi_1^k(T)} \int_{\varphi_1^k(T)}^{\varphi_1^k(T+U)} \left| \zeta \left(\frac{1}{2} + it \right) \right|^2 dt.
\end{aligned}$$

Hence, from (3.12) by (3.17), (3.18) the asymptotic formula (2.3) follows.

I would like to thank Michal Demetrian for helping me with the electronic version of this work.

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